ANALYSIS III MIDTERM EXAMINATION

Total marks: 40

(1) Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ be a bilinear function.
 (a) Prove that $lim_{(h,k)\to 0} \frac{|f(h,k)|}{|(h,k)|} = 0$.
 (b) Using this or otherwise, prove that $Df(a,b)(x,y) = f(a,y) + f(x,b)$. Here $Df(a,b)$ denotes the total derivative of f at the point $(a,b) \in \mathbb{R}^{n+m}$. (4+4=8 marks)

(2) (a) State the Inverse and the Implicit Function Theorems.
 (b) Show that the system of equations $x = u^4 - u + uv + v^2$, $y = cos(u) + sin(v)$, can be solved for (u,v) as a continuously differentiable function F of (x,y), in some neighborhood of $(0,0)$, in such a way that $(u,v) = (0,0)$ when $(x,y) = (0,1)$. What is the differential of F at $(0,1)$?
 (c) Can the equation $xz + yz + sin(x+y+z) = 0$ be solved, in a neighborhood of $(0,0,0)$ for z as a continuously differentiable function $z = g(x,y)$ of (x,y), with $g(0,0) = 0$? (6+4+4 = 14 marks)

(3) (a) Give an example of a bounded real valued function on a rectangle in $\mathbb{R}^d$ (for any d) which is not integrable.
 (b) Define a Jordan region, and also define a set with volume zero.
 (c) Prove that a bounded set $E \subset \mathbb{R}^d$ is a Jordan region, if and only if, the boundary of E has d-volume zero. (3+3+6 =12 marks)

(4) Let U be an open subset of $\mathbb{R}^2$, let $K \subset U$ be a compact set. Suppose that $f : U \to \mathbb{R}$ is a continuously differentiable function. Let $E := \{(x,y) \in K | f(x,y) = 0\}$. Suppose df is never zero on E. Show that E is a set of area (that is, 2-volume) zero in $\mathbb{R}^2$. (6 marks)